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On controllers of prime ideals in group algebras of torsion-free abelian groups of finite rank

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Abstract

Let kA be the group algebra of an abelian group A over a field k and let I be an ideal of kA. We say that a subgroup B of the group A controls the ideal I if $I = (I \cap kB)kA$. The intersection $c(I)$ of all subgroups of the group A controlling I is said to be the controller of the ideal I . The ideal I is said to be faithful if $I^{\dagger} = A \cap (1+I) = 1$. In the presented paper we develop some methods which allow us to study controllers of prime faithful ideals in group algebras of abelian groups of finite rank. The main idea is that the quotient ring kA/I by such an ideal I can be embedded as a domain $k[A]$ in a field F and the group A becomes a subgroup of the multiplicative group of the field F. It allows us to apply for studying of $k[A]$ some methods of the theory of fields such as Kummer theory and Dirichlet Unit theorem. In its tern properties of $k[A] \cong kA/I$ strongly depend on the properties of the ideal I. Using these methods, in particular, we obtain an independent proof of a Brookes theorem on controllers of prime ideals in the case, where the field k has characteristic zero.

Introduction

Let R be a ring, let G be a group and let I be a right ideal of the group ring RG. The ideal I is said to be faithful if $I^{\dagger} = G \cap (1 + I) = 1$. We say that a subgroup H of the group G controls the ideal I if

$$I = (I \cap kH)kG. \qquad (1)$$

The intersection $c(I)$ of all subgroups of the group G controlling I is said to be the controller of the ideal I. It is easy to note that a subgroup $B \leq G$ controls the ideal I if and only if $c(I) \leq B$.

Let H be a subgroup of the group G and let U be a right RH-module. Since the group ring RG can be considered as a left RH -module, we can define the tensor product $U \otimes_{RH} RG$ (see [5, 5.1]), which, by [5, 5.1, proposition 3],

is a right RG-module named the RG-module induced from the RH-module U. By [2, Chap. III, proposition 5.3], if M is an RG -module and $U \leq M$, then

$$M = U \otimes_{RH} RG \qquad (2)$$

if and only if

$$M = \oplus_{t \in T} Ut, \qquad (3)$$

where T - is a right transversal to subgroup H in G.

If a kG-module M of some representation φ of a group G over a field k is induced from some kH-module U, where H is a subgroup of the group G, then we say that the representation φ is induced from a representation of subgroup H, where U is the module of the representation ϕ.

Suppose that $M = aRG$ is a cyclic RG-module generated by a nonzero element $a \in M$. Put $I = Ann_{kG}(a)$ and let $U = akH$, where H is a subgroup of the group G. It is not difficult to note that, in these denotations, the equation (1) holds if and only if the equation (3) holds. Thus, in this case, the equations (1), (2), (3) mean the same.

If k is a subfield of a field f and A is a subset of f then $[f : k]$ is the dimension of the field f over k; by $k(A)$ we denote the field generated by k and A; by $k[A]$ we denote the domain generated by k and A. As usually, f^* denotes the multiplicative group of the field f. Let A be an abelian group then $t(A)$ denotes the torsion subgroup of A. If the group A is torsion then $\pi(A)$ is the set of prime divisors of orders of elements of A.

Let k be a field and let A be an abelian group of finite rank. In the presented paper we consider properties of controllers of faithful prime ideals in the group algebra kA. The main idea of our studying is that in the case, where P is a faithful prime ideal of kA, the quotient ring kA/P can be embedded as a domain $k[A]$ in a field F and, as the ideal P is faithful, the group A becomes a subgroup of the multiplicative group of the field F. It allows us to apply for studying of $k[A]$ some methods of the theory of fields such as Kummer theory and Dirichlet Unit theorem. In its tern properties of $k[A] \cong kA/P$ strongly depend on the properties of the ideal P. To prove equation (1) we prove equations (2), (3) for a kA-module $k[A]$.

In section 1 we consider relations between Kummer theory and equations (2) and (3) for modules over abelian groups. The main result of this section (theorem 1.3) can be considered as a generalization of [6, Chap. VIII, theorem 13] to the case of infinite dimensional extensions.

In section 2 we study properties of multiplicative groups of certain fields which are generated by a finite set and by roots from 1. The most important is that the multiplicative group of such a field is a direct product of Chernikov and free abelian groups (see proposition 2.3).

A combination of results of sections 1 and 2 allows us to study controllers of faithful prime ideals in group algebras of abelian groups of finite rank. For

the first time, such methods were applied in [9] (see [9, lemmas 2, 5]) where we proved the equation (3) for modules over abelian groups of finite rank. Then the methods were developed in [8, 10, 11, 12]. The main result (theorem 3.1) states that in the case where the field k is finitely generated of characteristic zero, the controller of any faithful prime ideal P of the group algebra kA of an abelian group A of finite rank with Chernikov torsion subgroup is finitely generated. In the case, where the group A is minimax, such a result follows from a Segal theorem [8, theorem 1.1]. An abelian group is said to be minimax if it has a finite series each of whose factor is either cyclic or quasi-cyclic.

In section 4 we obtain an independent proof of a Brookes theorem [1, theorem A] in the case of the field of characteristic zero (theorem 4.4). As it became known recently, the original proof of [1, theorem A] is incorrect. Moreover, there is an example which shows that in fact the theorem does not hold in the case of the field of positive characteristic. So, a new independent proof of the Brookes theorem in the case of the field of characteristic zero is quite topical.

1. Kummer theory and induced modules.

Let k be a subfield of a field f and let G be a subgroup of the multiplicative group f^* of the field f. Then the field $k(G)$ may be considered as a kG-module and the field k can be considered as a $k(G \cap k^*)$-module. Therefore, we can define the tensor product $k \otimes_{k(G\cap k^*)} kG$ and the equation $k(G) = k \otimes_{k(G\cap k^*)} kG$ means that $k(G) = \oplus_{t\in T} kt$, where T is a transversal to the subgroup $G \cap k^*$ in the group G. If $|G/G \cap k^*| = m < \infty$ the equation $k(G) = k \otimes_{k(G\cap k^*)} kG$ holds if and only if $[k(G) : k] = m$. The relations between $|G/G \cap k^*|$ and $[k(G) : k]$ were considered in Kummer theory (see [6, Chap. VIII, theorem 13]). So, we can hope that there are relations between induced modules over abelian groups and Kummer theory. These relations are studied in this section.

Lemma 1.1. *Let k be a subfield of a field f and let G be a subgroup of f^* such that $k^* \leq G$, k contains a primitive root from 1 of degree 4 if the quotient group G/k^* has an element of order 4 , the quotient group G/k^* is torsion, for any $p \in (\pi(t(G)) \cap \pi(G/k^*))$ the field k contains a primitive root from 1 of degree p and $\operatorname{char} k \notin \pi(G/k^*)$. Let $g \in G\backslash k^*$ and let $\bar{g}$ be the image of g in the quotient group G/k^*. Suppose that $|\bar{g}| = t$ then $[k(g) : k] = t$.*

Proof. Let $g^t = a \in k^*$. Suppose that $a = 1$ then $g \in t(G)$ and hence all prime divisors of t belong to the set $\pi(t(G)) \cap \pi(G/k^*)$. But then for any prime divisor p of t the field k contains a primitive root from 1 of degree p, and it easily implies that $a \neq 1$.

Suppose that for some prime divisor p of t there is an element $b \in k$, such that $b^p = a$. Then $(g^m b^{-1})^p = 1$, where $m = t/p$. Evidently, $p \in (\pi(t(G)) \cap \pi(G/k^*))$ and hence k contains a primitive root from 1 of degree p. Therefore, $g^m b^{-1} \in k^*$ and, as $b \in k^*$, we have $g^m \in k^*$. But it is impossible

because $|\bar{g}| = t$.

Suppose that 4 divides t and $a \in -4k^4$ then there is an element $b \in k$ such that $a = -4b^4$ and the quotient group G/k^* has an element of order 4. As the quotient group G/k^* has an element of order 4, the field k contains a primitive root from 1 of degree 4. Let $m = t/4$ and $h = g^m$ then $h^4 = -4b^4$ and hence $h^2 = (\pm 2i)b^2$, where i is a primitive root from 1 of degree 4. Since the field k contains i, we have $h^2 = g^{2m} \in k^*$. But it is impossible because $|\bar{g}| = t$.

Thus, $a \notin k^p$ for any prime divisor p of t and $a \notin -4k^k$ if 4 divides t. Then it follows from [6, Chap. VIII, theorem 16] that the element g is a root of an irreducible polynomial $X^t - a$ over the field k and, by [6, Chap. VII, proposition 3], $[d : k] = t$.

Lemma 1.2. *Let k be a subfield of a field f and let G be a subgroup of f^* such that $k^* \leq G$, k contains a primitive root from 1 of degree 4 if the quotient group G/k^* has an element of order 4, the quotient group G/k^* is torsion, for any $p \in (\pi(t(G)) \cap \pi(G/k^*))$ the field k contains a primitive root from 1 of degree p and $\mathrm{char}\, k \notin \pi(G/k^*)$. Suppose that $|G/k^*| = n$, where $n = p^m$ and p is a prime number. Then $[k(G) : k] = p^m = n$.*

Proof. Suppose that the field k contains a primitive root ξ from 1 of degree p. The proof is by induction on m.

Let g be an element of G such that $|\bar{g}| = p$, where $\bar{g}$ is the image of g in the quotient group G/k^*. Let $d = k(g)$ then, by lemma 1.1, $[d : k] = p$. Suppose that there is an element $h \in G \cap d^*$ such that $\left|\bar{h}\right| = p^2$, where $\bar{h}$ is the image of h in the quotient group G/k^*.

By lemma 1.1, $[k(h) : k] = p^2$ and, as $k(h) \leq k(g)$ and $[k(g) : k] = p$, we obtain a contradiction with [6, Chap. VII, proposition 2]. Thus, we can assume that the quotient group $(d^* \cap G)/k^*$ is elementary abelian. Put $d^* \cap G = L$, then $k^* \leq L$ and , evidently, $d = k(L)$. By [6, Chap. VIII, theorem 13], $p = [d : k] = |L^p/(k^*)^p|$ and, as the field k contains a primitive root from 1 of degree p, it is not difficult to show that $|L/k^*| = p$. Then $G \cap d^* = k^* < g >$ and hence $|G/d^*| = p^{m-1}$. Therefore, by the induction hypothesis, $[d(G) : d] = p^{m-1}$. Evidently, $d(G) = k(G)$ and, as $[d : k] = p$, it follows from [6, Chap. VII, Proposition 2] that $[k(G) : k] = p^m = n$.

Suppose that the field k does not contain ξ. Put $d = k(\xi)$ then $[d : k] = r < p$ because the cyclotomic polynomial for ξ has degree $p - 1$.

Now, we show that $G \cap d^* = k^*$. Suppose that there is an element $g \in (G \cap d^*)\backslash k^*$. Evidently, there is no harm in assuming that $|\bar{g}| = p$, where $\bar{g}$ is the image of g in the quotient group $(G \cap d^*)/k^*$. Then, by lemma 1.1, $[k(g) : k] = p$ but, as $k(g) \leq d$, we obtain a contradiction with $[d : k] = r < p$. Thus, $G \cap d^* = k^*$ and hence $Gd^*/d^* \cong G/(G \cap d^*) = G/k^*$. Therefore, $|G/(d^* \cap G)| = n = p^m$. Then, as it was shown above, $[d(G) : d] = p^m$ and hence, by [6, Chap. VII, Proposition 2], $[d(G) : k] = p^m r$, where $[d : k] = r < p$. Since $d(G) = k(G)(\xi)$, we have $[d(G) : k(G)] \leq r$. Then it follows from [6, Chap. VII, proposition 2] and the equation $[d(G) : k] = p^m r$

that $[k(G):k] \geq n$. But the relation $[k(G):k] > n$ is impossible, because $[k(G):k] \leq |G/k^*| = n$. Thus, $[k(G):k] = n$.

Theorem 1.3. *Let k be a subfield of a field f and let G be a subgroup of f^* such that $k^* \leq G$, k contains a primitive root from 1 of degree 4 if the quotient group G/k^* has an element of order 4, the quotient group G/k^* is torsion, for any $p \in (\pi(t(G)) \cap \pi(G/k^*))$ the field k contains a primitive root from 1 of degree p and $char k \notin \pi(G/k^*)$. Then $k(G) = k \otimes_{k(k^*)} kG = \oplus_{t\in T} kt$, where T is a transversal to k^* In G.*

Proof. Since the quotient group G/k^* is locally finite, it is sufficient to show that for any subgroup $H \leq G$ such that $k^* \leq H$ and $|H/k^*| < \infty$ the equation $k(H) = k \otimes_{k(k^*)} kH$ holds. Thus, there is no harm in assuming that $|G/k^*| = n < \infty$ and $char k$ does not divide n. Since the equation $k(G) = k \otimes_{k(k^*)} kG$ means that $k(G) = \oplus_{t\in T} kt$, where T is a transversal to k^* in the group G, it is sufficient to show that $[k(G):k] = n$. We use the induction on n.

Let p be the smallest prime divisor of n and let N/k^* be the Sylow p-subgroup of the quotient group G/k^*. Suppose that $|N/k^*| = t$. Let $d = k(N)$, then by lemma 1.2, $[d:k] = t$.

Now, we show that $d^* \cap G = N$. Suppose, that there is an element $g \in (d^* \cap G)\backslash N$. Evidently, there is no harm in assuming that $|\bar{g}| = p'$, where $\bar{g}$ is the image of the element g in the quotient group $G/(k^* \cap G)$ and p' is a prime number such that $p < p'$. Then, by lemma 1.1, $[k(g):k] = p'$ and, as $k(g) \leq d$, by [6, Chap. VII, proposition 2], p' divides t but it is impossible because $p \neq p'$. Thus, $d^* \cap G = N$ and hence $Gd^*/d^* \cong G/(d^* \cap G) = G/N$. So, we can conclude that $|Gd^*/d^*| = |G/N| = n/t$. Then by the induction hypothesis, $[d(G):d] = n/t$ and, as $[d:k] = t$, by [6, Chap. VII, proposition 2], we have $[d(G):k] = n$. Finally, it is easy to see that $d(G) = k(G)$ and the assertion follows.

Corollary 1.4. *Let k be a subfield of a field f and let G be a subgroup of f^* such that k contains a primitive root from 1 of degree 4 if the quotient group Gk^*/k^* has an element of order 4, the quotient group Gk^*/k^* is torsion, for any $p \in (\pi(t(Gk^*)) \cap \pi(Gk^*/k^*))$ the field k contains a primitive root from 1 of degree p and $char k \notin \pi(G/k^*)$. Then $k(G) = k \otimes_{k(k^* \cap G)} kG = \oplus_{t\in T} kt$, where T is a transversal to $k^* \cap G$ in G.*

Proof. To prove the corollary, it is sufficient to apply the above theorem to the field k and the group Gk^*.

2. On multiplicative groups of certain fields.

We will say that a field k is regular if it is countable and the multiplicative group of the field k is a direct product of a torsion group and a free abelian group.

Lemma 2.1. *Let $f = k(S)$ be a transcendent extension of a field k, where*

S is a finite set of elements of the field f. If the field k is regular then so is the field f.

Proof. Since the set S is finite, so is the transcendent degree of the field f over the subfield k. Let $z \notin k$ and let $z, z_1, ...z_n$ be a maximal system of algebraically independent over k elements of the field f . Put $f_1 = k(z, z_1, ..., z_n)$ then it is not difficult to show that $f_1^* = k^* \times N$, where N is a countable free abelian group and hence, as the field k is regular, so is the field f_1. As $f \geq f_1 \geq k$ and $f = k(S)$, we can conclude that $f = f_1(S)$. Then, as the field f is an algebraic extension of the field f_1, we see that $[f : f_1]$ is finite and hence we can put $[f : f_1] = m < \infty$.

Let $\varphi_z : f^* \to f_1^*$, be a homomorphism which maps each element of the multiplicative group f^* to its regular norm over the field f_1. Then, evidently, $\varphi_z(z) = z^m$.

Suppose that $z \in t(f^*/Ker\varphi_z)$ then there is a positive integer r such that $z^r \in Ker\varphi_z$ and hence $1 = \varphi_z(z^r) = (\varphi_z(z))^r = z^{mr}$. Therefore, $z \in t(f^*)$. On the other hand, as the field f is a transcendent extension of the field k, the quotient group f^*/k^* is torsion-free and , as $z \notin k$, we see that $z \notin t(f^*)$. Thus, a contradiction is obtained and hence $z \notin t(f^*/Ker\varphi_z)$. Since $f^*/Ker\varphi_z \cong \varphi_z(f^*) \leq f_1^*$ and the field f_1 is regular, it is not difficult to show that the quotient group f^*/f_x is free abelian , where f_x is a subgroup of f^* and $f_x/Ker\varphi_z = t(f^*/Ker\varphi_z)$.

Thus, for any element $z \notin f \backslash k$ there exists a subgroup f_z of the group f^* such that $z \notin f_z$ and the quotient group f^*/f_z is a free abelian. Put $T = \bigcap_{z \in f \backslash k} f_z$, evidently $T \leq k^*$ and the quotient group f^*/T is a residually free abelian and, as the group f^* is countable, it implies that the quotient group f^*/T is free abelian. Since $T \leq k^*$, we see that T is a direct product of a torsion and a free abelian groups and hence the field f is regular.

Let k be a field, the quotient group $\bar{k} = k^*/t(k^*)$ is said to be the reduced multiplicative group of the field k.

Lemma 2.2.*Let f be a finite extension of a field k such that $[f : k] = n$. Suppose that the field f is regular. Then $\bar{f} = F \times (\bar{f} \cap \bar{k}^{1/n})$, where F is a countable free abelian group.*

Proof. Let $\varphi : f^* \to k^*$ be a homomorphism given by $\varphi : x \mapsto |x|$, where $|x|$ is the regular norm of an element $x \in f$ over k. It is easy to note that φ induces a homomorphism $\bar{\varphi} : \bar{f} \to \bar{k}$. It follows from the definition of the homomorphism $\bar{\varphi}$ that $\bar{\varphi}(x) = x^n$ for any $x \in \bar{k}$. Since the field f is regular, we see that the groups $\bar{k}$ and $Ker\bar{\varphi}$ are free abelian. Put $A = Ker\bar{\varphi}$, since $\bar{\varphi}(x) = x^n$ for any $\bar{x} \in k$, we can conclude that $1 = A \cap \bar{k}$ and hence $1 = A \cap (\bar{f} \cap \bar{k}^{1/n})$.

Let $x \in \bar{f}$ and $\bar{\varphi}(x) = y \in \bar{k}$ then $\bar{\varphi}(x^n y^{-1}) = \bar{\varphi}(x^n)\bar{\varphi}(y^{-1}) = \bar{\varphi}(x)^n (y^{-1})^n = y^n (y^{-1})^n = 1$. Thus, for any element $x \in \bar{f}$ there is an element $y \in \bar{k}$ such that $x^n y^{-1} \in A$. Therefore, $(\bar{f})^n \leq \bar{k} \times A$ and hence the quotient group $R = \bar{f}/(\bar{f} \cap \bar{k}^{1/n})$ is torsion free. Since $1 = A \cap (\bar{f} \cap \bar{k}^{1/n})$ and the subgroup A

is free abelian, we can conclude that the quotient group R has a free abelian subgroup $T = (A(\bar{f} \cap \bar{k}^{1/n}))/(\bar{f} \cap \bar{k}^{1/n})$ such that $R^n \leq T$. Then, as the group R is torsion free, it easily implies that the group R is free abelian. Thus, the quotient group $R = \bar{f}/(\bar{f} \cap \bar{k}^{1/n})$ is free abelian and hence there is a free abelian subgroup $F \leq \bar{f}$ such that $\bar{f} = F \times (\bar{f} \cap \bar{k}^{1/n})$.

Proposition 2.3. *Let f be a field of characteristic zero generated by a finite set S and by all roots from 1 of degree p^n, where n are positive integers, $p \in \pi$ and π is a finite set of prime numbers. Then the set $\pi(t(f^*))$ is finite and the field f is regular, $f^* = T \times F$, where F is a free abelian group and T:*
(i) is a finite group if the field f is finitely generated (that is if $\pi = \emptyset$);
(ii) is a locally cyclic Chernikov group.

Proof. Let k be a subfield of f generated by all roots from 1 of degree p^n, where n are positive integer and $p \in \pi$, then $f = k(S)$. If $\pi = \emptyset$ then k is the field of rational numbers. Let k_1 be the maximal algebraic extension of k in f. Since $|S| < \infty$, we see that $[k_1 : k] < \infty$ and f is a finitely generated transcendent extension of k_1. As f is a finitely generated transcendent extension of k_1, we see that $t(f^*) = t(k_1^*)$ and it follows from lemma 2.1 that it is sufficient to show that the field k_1 is regular. So, we can assume that the elements of S are algebraic over k and hence $[f : k] = r < \infty$.

(i) In this case, f is a finitely generated algebraic field of characteristic zero and hence f is a finite extension of the rational number field. Then, be the Dirichlet Unit theorem (see [4, Chap. I, theorem 13.12]), the group U of unites of the ring R of algebraic integers of the field f is finitely generated. There exists a homomorphism φ of the group f^* into the free abelian group $I(R)$ of fractional ideals of R which maps each element $a \in f^*$ into the fractional ideal generated by the element a (see [4, Chap. I, section 4]) and $Ker\varphi = U$. Then, by the theorem on homomorphism, the quotient group f^*/U is free abelian and, as the group U is finitely generated, it easily implies that $f^* = T \times F$, where F is a free abelian group and T is a finite group.

(ii) At first, we show that the set $\pi(t(f^*))$ is finite and hence the group $t(f^*)$ is Chernikov. Suppose that the field f contains a primitive root ξ from 1 of degree $q \notin \pi$. It is well known that the cyclotomic polynomial for ξ is irreducible over k and hence $[k(\xi) : k] = q - 1$. Then $q - 1 \leq r$ and evidently the set of all such q is finite and, as the set π is finite, so is $\pi(t(f^*))$. Therefore, $T = t(f^*)$ is a Chernikov locally cyclic group.

Now, we show that the field f is regular. Let D be the set of all roots from 1 of degree p^2, where $p \in \pi$, then the set D is finite. Let h be a subfield of f generated by $(S \cup D)$. Then, by (i), h^* is a direct product of a finite and a free abelian groups. Evidently, the field f has an infinite series $\{h_i | i \in I\}$ of subfields such that $h = h_1$, $h_i \leq h_{i+1}$, $\cup_{i \in I} h_i = f$ and $h_{i+1} = h_i(\zeta_i)$, where ζ_i is a root from 1 of degree p^n for some $p \in \pi$ and positive integer n. We also can assume that $\zeta_i^p \in h_i$ then it follows from lemma 1.1 that $[h_{i+1} : h_i] = p$ if $h_i \neq h_{i+1}$. Evidently, $\bar{h}_i \leq \bar{h}_{i+1}$, $\cup_{i \in I} \bar{h}_i = \bar{f}$ and it follows from (i) that each group $\bar{h}_i$ is free abelian. Then it is sufficient to show that the quotient group

$\bar{h}_{i+1}/\bar{h}_i$ is free abelian for each i.

By lemma 2.2, $\bar{h}_{i+1} = F_i \times (\bar{h}_{i+1} \cap \bar{h}_i^{1/p})$, where F_i is a countable free abelian group and it is sufficient to show that $\bar{h}_i = \bar{h}_{i+1} \cap \bar{h}_i^{1/p}$. Suppose that $\bar{h}_i \neq \bar{h}_{i+1} \cap \bar{h}_i^{1/p}$ then there is an element $a \in \bar{h}_{i+1} \backslash \bar{h}_i$ such that $a^p \in \bar{h}_i$ and hence there is an element $b \in h_{i+1} \backslash h_i$ such that $b^p \in h_i$ and $b \notin \langle \zeta_i \rangle$. It easily implies that the group h_{i+1}^* has a subgroup G such that $h_i^* \leq G$ and G/h_i^* is an elementary abelian p-group of order p^2. Therefore, as h contains a primitive root from 1 of degree p^2, it follows from lemma 1.2 that $[h_i(G) : h_i] = p^2$ but it is impossible because $h_i^*(G) \leq h_{i+1}^*$ and $[h_{i+1} : h_i] = p$.

Thus, the field f is regular and hence $f^* = T \times F$, where T is a locally cyclic Chernikov group and F is a free abelian group.

3. Controllers in prime ideals of abelian groups of finite rank.

Let A be an abelian group and let B be a subgroup of A. The set $is_A(B)$ of elements $a \in A$ such that $a^n \in B$, fore some positive integer n, is a subgroup of A which is said to be the isolator of the subgroup B in the group A. The subgroup B is said to be dense if $is_A(B) = A$ and the subgroup B is said to be isolated if $is_A(B) = B$.

Theorem 3.1. *Let k be a finitely generated field of characteristic zero, let G be an abelian group of finite rank such that the torsion subgroup $t(G)$ is Chernikov and let P be a prime faithful ideal of kG. Then the controller of P is finitely generated.*

Proof. Let M be the field of fraction the domain kG/P, then $M = k(G)$ and G is a subgroup of the multiplicative group of M. Let h be the algebraic closure of the field M. Let $d = k(H, i)$, where H is a finitely generated dense subgroup of G which contains elements of order p for each $p \in \tau = \pi(t(G))$, and i is a primitive root of degree 4 from 1. By proposition 2.3(i), $d^* = T \times L$, where T is a finite group and L is a free abelian group. It implies, that $G \cap d^*$ is a finitely generated dense subgroup of G and changing H by $G \cap d^*$ we can assume that $H = G \cap d^*$.

Let $D = Gd^*$ then $D/d^* = Gd^*/d^* = G/(G \cap d^*) = G/H$ and hence the quotient group D/d^* is torsion, besides d contains a primitive root of degree 4 from 1. So, it would be possible to apply theorem 1.3 but there may be a situation that not for any $p \in (\pi(t(D)) \cap \pi(D/d^*))$ the field d contains a primitive root from 1 of degree p. However, as $D = Gd^*$, where $\pi(t(G)) \subseteq \pi(t(d^*))$ and d^* is an almost free abelian group, the above situation may be possible only for $p \in \pi = \pi((is_{\bar{d}}\bar{H})/\bar{H})$, where $\bar{d} = d^*/t(d^*)$ and $\bar{H} = Ht(d^*)/t(d^*)$. Since the group $\bar{d}$ is free abelian, it is easy to note that the set π is finite. Let p be the biggest prime from π and let ω be the set of all primes $q \leq p$. Let ξ be a primitive root from 1 of degree p, let $d_1 = d(\xi)$ and let $H_1 = G \cap d_1^*$. By lemma 2.2, $\bar{d_1} = R \times (\bar{d_1} \cap \bar{d}^{1/n})$, where R is a free abelian group and $n = [d_1 : d] < p$, and hence $\pi((is_{\bar{d_1}}\bar{H}_1)/\bar{H}_1) \subseteq \varpi \backslash \{p\}$, where $\bar{d_1} = d_1^*/t(d_1^*)$ and $\bar{H}_1 = (G \cap d_1^*)t(d_1^*)/t(d_1^*)$. Thus, after several steps,

adding primitive roots from 1, we obtain a field f such that the quotient group F/f^* is torsion, where $F = Gf^*$, $i \in f$ and for any $p \in (\pi(t(F)) \cap \pi(F/f^*))$ the field f contains a primitive root from 1 of degree p. Therefore, by theorem 1.3, $f(F) = \oplus_{t \in T} ft$, where T is a transversal to f^* in F. Since $F = Gf^*$ we see that $f(G) = \oplus_{t \in T} ft$, and T can be chosen as a transversal to $L = f^* \cap G$ in G. Therefore, $k(G) = \oplus_{t \in T} k(L)t$ and hence $k[G] = \oplus_{t \in T} k[L]t$, where T is a transversal to L in G. Evidently, it implies that $P = (P \cap kL)kG$ and, as the field f is finitely generated, it follows from proposition 1.3(i) that so is the subgroup L.

Corollary 3.2. *Let k be a finitely generated field of characteristic zero and let G be an abelian group of finite rank such that the torsion subgroup $t(G)$ is Chernikov. Then any faithful irreducible representation of the group G over the field k is induced from some finitely generated subgroup of G.*

Proof. Let M be a module of a faithful irreducible representation of the group G over the field k. Then M is a simple kG-module and hence $M \cong kG/P$, where $P = Ann_{kG}(a)$ for some nonzero element $a \in M$ is a faithful maximal ideal of kG. By the above theorem, the ideal P is controlled by a finitely generated subgroup $H \leq G$. It means that $M = \oplus_{t \in T} Ut$, where $U = kH/Ann_{kH}(a)$ and T is a transversal to H in G, and hence $M = U \otimes_{kH} kG$.

Theorem 3.3. *Let k be a finitely generated field of characteristic zero, let A be a torsion-free abelian minimax group and let P be a faithful prime ideal of the group algebra kA. Let $R = kA/P$ and let h be the field of fractions of R then $A \leq h^*$. Let C be a finitely generated dense subgroup of A and let τ be the set of all roots from 1 of degree p^n for all $p \in \pi(A/C)$ and all positive integer n in addition with a primitive root of degree 4 from 1 . Let f be a field obtained by addition of all roots from the set τ to the field h and let s be a subfield of f generated by all roots from τ and by the subgroup C and let $B = A \cap s^*$. Let D be a dense subgroup of C, let d be a subfield of f generated by D and by $t(s^*)$ and let $H = A \cap d^*$. Then :*
(i) B and H are finitely generated dense subgroups of A, besides $B \geq H$;
(ii) B and H control the ideal P.

Proof. (i) By proposition 2.3(ii), the field s is regular and hence so is its subfield d. It easily implies that B and H are finitely generated dense subgroups of A and the relation $B \geq H$ is evident.

(ii) Put $F = As^*$. Since B is a dense subgroup of A, we see that the quotient group F/s^* is torsion. By the definition of τ, s contains a primitive root from 1 of degree 4 and a primitive root of degree p for any $p \in \pi(F/s^*)$. So, we can conclude that F and s meet all conditions of theorem 1.3 and hence $s(A) = s \otimes_{s(s^*)} sA$. Therefore, $k[A] = k[B] \otimes_{kB} kA$ and hence $P = (P \cap kB)kA$.

Put $L = Ad^*$. Since H is a dense subgroup of A, we see that L/d^* is a torsion group. As $d \leq s$ and $t(s^*) \leq d$, we can conclude that $t(s^*) = t(d^*)$ and hence d contains a primitive root from 1 of degree 4.

Now, we show that for any $p \in (\pi(t(L)) \cap \pi(L/d^*))$ the field d contains a primitive root from 1 of degree p. Evidently, $L/d^* = Ad^*/d^* \cong A/(A \cap d^*) = A/H$ and the group A has a series of subgroups $A \geq B \geq H$. Since $\pi(A/B) \subseteq \tau$ and $\tau \subseteq \pi(t(s^*)) = \pi(t(d^*))$, it is sufficient to consider the case where $p \in (\pi(t(Bd^*)) \cap \pi(B/H))$. But $Bd^* \leq s^*$ and, as $t(s^*) = t(d^*)$, we can conclude that $p \in \pi(t(d^*))$. Thus, L and d meet all conditions of theorem 1.3 and the above arguments show that the subgroup H controls the ideal P.

4. On controllers of standardized prime faithful ideals in group algebras of abelian groups of finite rank.

Let k be a field, let A be a torsion-free abelian group of finite rank acted by a group Γ and let I be an ideal of a group algebra kA. The set $\Delta_\Gamma(A)$ of elements of A which have finite orbits under action of the group Γ is a subgroup of the group A. The subgroup $N_\Gamma(I) \leq \Gamma$ of elements $\gamma \in \Gamma$ such that $I = I^\gamma$ is said to be the normalizer of the ideal I in the group Γ. The subgroup $S_\Gamma(I) \leq \Gamma$ of elements $\gamma \in \Gamma$ such that $I \cap kB = I^\gamma \cap kB$ for some finitely generated dense subgroup B of A is said to be the standardiser of the ideal I in the group Γ. Evidently, $N_\Gamma(I) \leq S_\Gamma(I)$. Theorem 4.4 states that if the field k has characteristic zero and P is a prime faithful ideal of the group algebra kA such that $S_\Gamma(P) = \Gamma$ then P is controlled by $\Delta_\Gamma(A)$. In the case, where the group A is finitely generated and $|\Gamma : N_\Gamma(P)| < \infty$, such a result was proved by Roseblade in [7, theorem D]. As lemma 4.2 shows, it is sufficient to consider the situation where the field k is finitely generated, the group A is minimax and the group Γ is cyclic. Lemma 4.3 shows that it is sufficient to prove that the ideal P is controlled by a finitely generated Γ-invariant subgroup of A. So, if the ideal P is Γ-invariant, the result would follow immediately from theorem 3.1 or from [8, theorem 1.1], because the controller of a Γ-invariant ideal is a Γ-invariant subgroup. However, the condition $S_\Gamma(P) = \Gamma$ is much more general than $N_\Gamma(P) = \Gamma$ and does not mean that the ideal P is Γ-invariant. This circumstance creates the main difficulties which are conquered in the proof of theorem 4.4.

Lemma 4.1. *Let k be a field, let A be a torsion-free abelian group of finite rank acted by a group Γ and let P be a faithful prime ideal of the group algebra kA. Then:*
(i) $r(c(P)) = r(c(kB \cap P))$ for any dense subgroup B of A;
(ii) if $S_\Gamma(P) = \Gamma$ then $is_A c(P)$ is a Γ-invariant subgroup of A.

Proof. (i) Since $c(kB \cap P) \leq c(P) \cap B$, there is no harm in assuming that $is_A c(P) = A$. Suppose that $r(c(P)) > r(c(kB \cap P))$ then there is an isolated subgroup D of A such that $c(P \cap kB) \leq D \cap B$ and $r(A/D) = 1$. Let C be a maximal isolated subgroup of D such that $P \cap kC = 0$. Since $c(P)$ is a dense subgroup of A, we can conclude that C is a maximal subgroup of A such that $kC \cap P = 0$. Then the transcendence degree over k of the field of fraction k_1 of the domain kA/P is $r(C)$. On the other hand, as

$k(C \cap B) \cap P = 0$ and $c(P \cap kB) \leq D \cap B$, we see that the transcendence degree of the field of fractions k_2 of the domain $kB/(kB \cap P)$ is at least $r(C) + 1$. But it is impossible because A/B is a torsion group and hence k_1 is an algebraic extension of k_2.

(ii) By the definition of $S_\Gamma(P)$, for any element $\gamma \in \Gamma$ there is a finitely generated dense subgroup B of A such that $kB \cap P = kB \cap P^\gamma$. Therefore $c(kB \cap P) \leq c(P) \cap c(P)^\gamma$, and by (i), $r(c(P)) = r(c(kB \cap P)) = r(c(P)^\gamma)$. It easily implies that $c(P) \cap c(P)^\gamma$ is a dense subgroup in $c(P)$ and $c(P)^\gamma$. Therefore, $is_A c(P) = is_A c(P)^\gamma$.

Lemma 4.2. *Suppose that there exist a field k, a torsion-free abelian group A of finite rank acted by a group Γ and a faithful prime ideal P of kA which is not controlled by $\Delta_\Gamma(A)$ and such that $S_\Gamma(P) = \Gamma$. Then there exist a finitely generated subfield $k_1 \leq k$, a minimax subgroup $A_1 \leq A$ acted by a cyclic subgroup $\langle\gamma\rangle \leq \Gamma$ and a faithful prime ideal $P_1 = P \cap k_1A_1$ of k_1A_1 which is not controlled by $\Delta_{\langle\gamma\rangle}(A_1)$ and such that $S_{\langle\gamma\rangle}(P) = \langle\gamma\rangle$.*

Proof. At first, we show that there is no harm in assuming that the field K is finitely generated. Suppose that P is not controlled by $\Delta_\Gamma(A)$ then there is an element $\alpha \in P \setminus ((P \cap k\Delta_\Gamma(A))kA)$. Let k_1 be a subfield of k generated by coefficients of α and let $P_1 = P \cap k_1A$ then $\alpha \in P_1 \setminus ((P_1 \cap k_1\Delta_\Gamma(A))k_1A)$ and hence P_1 is not controlled by $\Delta_\Gamma(A)$. It is easy to note that P_1 is a prime faithful ideal of k_1A and $S_\Gamma(P_1) = \Gamma$. Thus, it is sufficient to consider the case where the field k is finitely generated.

Let $c(P)$ be the controller of P, by lemma 4.1(ii), $is_A c(P)$ is a Γ-invariant subgroup of A. Then there is no harm in assuming that $is_A c(P) = A$. We also assume that Γ acts on the group A faithfully, that is $C_\Gamma(A) = 1$.

Suppose that $c(P)$ is not contained in $\Delta_\Gamma(A)$. Let F be a free dense subgroup of $c(P)$ with free generators $\{a_i | i = 1, ..., n\}$.

Suppose that for any element $\gamma \in \Gamma$ we have $\left|\langle\gamma\rangle / C_{\langle\gamma\rangle}(a_i)\right| < \infty$. Let $C_\gamma = \cap_{i=1}^n C_{\langle\gamma\rangle}(a_i)$ then $|\langle\gamma\rangle / C_\gamma| < \infty$ and it easy to see that C_γ centralizes A. Since Γ acts on the group A faithfully, it implies that the group Γ is torsion. But Γ is a linear group over the field of rational numbers and hence the group Γ is finite. Then, evidently, $\Delta_\Gamma(A) = A$ and a contradiction is obtained.

Thus, if $c(P)$ is not contained in $\Delta_\Gamma(A)$, then there are an element $a \in c(P)$ and an element $\gamma \in \Gamma$ such that the group $\langle\gamma\rangle$ is infinite cyclic and $C_{\langle\gamma\rangle}(a) = 1$. Then we can replace Γ by $\langle\gamma\rangle$ because $c(P)$ is not contained in $\Delta_{\langle\gamma\rangle}(A)$. Let A_1 be a dense subgroup of A which is finitely generated as a $\langle\gamma\rangle$-module and which contains the element a. Then, by [3, lemma 5.2], the subgroup A_1 is minimax.

Put $P_1 = P \cap kA_1$, by lemma 4.1(i), $c(P_1)$ is a dense subgroup ofA_1. If $c(P_1)$ is contained in $\Delta_\Gamma(A_1)$ then $\Delta_\Gamma(A_1)$ is a dense subgroup of A_1 and hence $\Delta_\Gamma(A_1) = A_1$. But it is impossible because $a \notin \Delta_\Gamma(A_1)$. Thus, $c(P_1)$ is not contained in $\Delta_\Gamma(A_1)$.

Lemma 4.3. *Let k be a field of characteristic zero, let A be a torsion-free finitely generated abelian group acted by a cyclic group $\Gamma = \langle g \rangle$ and let P be a faithful prime ideal of the group algebra kA. Suppose that $S_\Gamma(P) = \Gamma$ then P is controlled by $\Delta_\Gamma(A)$.*

Proof. Since $S_\Gamma(P) = \Gamma$, there is a subgroup B of finite index in A such that $P \cap kB = P^\gamma \cap kB$. As $A^n \leq B$ fore some positive integer n, we can assume that the subgroup B is Γ-invariant and it implies that the ideal $P_1 = P \cap kB$ is $\langle \gamma \rangle$-invariant. Since $|A : B| < \infty$, there is only finite set of ideals X of kA such that $P_1 = X \cap kB$. Then, as the ideal P_1 is $\langle \gamma \rangle$-invariant and $\Gamma = \langle g \rangle$, it easily implies that $|\Gamma : N_\Gamma(P)| < \infty$ and the assertion follows from [7, theorem D].

Theorem 4.4. *Let k be a field of characteristic zero, let A be a torsion-free abelian group of finite rank acted by a group Γ and let P be a faithful prime ideal of the group algebra kA. Suppose that $S_\Gamma(P) = \Gamma$ then P is controlled by $\Delta_\Gamma(A)$.*

Proof. By lemma 4.2, it is sufficient to consider the case where the field k is finitely generated, the group A is minimax and the group Γ is cyclic.

By lemma 4.1(i), $is_A c(P)$ is a Γ-invariant subgroup of A and changing A by $is_A c(P)$, we can assume that $c(P)$ is a dense subgroup of A. Then it follows from theorem 3.1 that $c(P)$ is a finitely generated dense subgroup of A.

Since the group A is minimax, the set $\pi(A/c(P))$ is finite. Let τ be the set of all roots from 1 of degree p^n for all $p \in \pi(A/c(P))$ and all positive integer n in addition with a primitive root from 1 of degree 4.

Put $C = c(P)$. Since $S_\Gamma(P) = \Gamma$, for any $\gamma \in \Gamma$ there is a finitely generated dense subgroup D_γ of A such that $kD_\gamma \cap P = kD_\gamma \cap P^\gamma$. Evidently, we can assume that $D_\gamma \leq C \cap C^\gamma$.

Let $R = kA/P$ and let h be the field of fractions of R, then $A \leq h^*$. Let f be a field obtained by addition to the field h of all roots from the set τ and let s be a subfield of f generated by all roots from the set τ and the subgroup C. Let $B = A \cap s^*$ then, by theorem 3.3, B is a finitely generated dense subgroup of A which controls P. Let d be a subfield of f generated by D_γ and $t(s^*)$ and let $H = A \cap d^*$ then, by theorem 3.3, $c(P) \leq H \leq B$.

Let $R_\gamma = kA/P^\gamma$ and let h_γ be the field of fractions of R_γ. Let f_γ be a field obtained by addition to the field h_γ all roots from the set τ and let s_γ be a subfield of f_γ generated by all roots from the set τ and the subgroup C^γ. Let d_γ be a subfield of f_γ generated by D_γ and $t(s_\gamma^*)$ and let $H_\gamma = A \cap (d_\gamma^*)$ then, by theorem 3.3, $C^\gamma = c(P^\gamma) \leq H_\gamma$.

Since $(kA/P)^\gamma = kA/P^\gamma$, we can conclude that the fields f and f_γ are isomorphic under the isomorphism φ induced by action of γ. Since $\varphi(C) = C^\gamma$, we see that the fields s and s_γ are isomorphic and hence $t(s^*) \cong t(s_\gamma^*)$.

Let K_1 be a subring of f generated by k and D_γ, and let K_2 be a subring of f_γ generated by k and D_γ. Since $kD_\gamma \cap P = kD_\gamma \cap P^\gamma$, we can conclude that $K_1 = K_2 = K$. Then, as $t(s^*) \cong t(s_\gamma^*)$, the fields d and d_γ are the fields

of decomposition of the same set Ω of polynomials over the field of fractions of the domain K. More precisely, Ω is the set of cyclotomic polynomials for roots from 1 which belong to the set $t(s^*) \cong t(s^*_\gamma)$. Therefore, by [6, Chap. VII, theorem 3], there is an isomorphism ψ between the fields d and d_γ which centralizes the elements of K and we can assume that $K \leq d \cap d_\gamma$.

Put $\pi = \pi(A/C)$ then $\pi = \pi(A/C^\gamma)$. As $C \leq H$ and $C^\gamma \leq H_\gamma$, we can conclude that A/H and A/H_γ are π-groups.

Suppose that $H \neq H_\gamma$ then either $H \backslash H_\gamma \neq \emptyset$ or $H_\gamma \backslash H \neq \emptyset$. Suppose that $H \backslash H_\gamma \neq \emptyset$ then $H/(H \cap H_\gamma)$ is a nontrivial π-group and hence the Hall π-subgroup of the quotient group H/D_γ is not contained in $(H \cap H_\gamma)/D_\gamma$. Therefore, there exists an element $a \in H \backslash H_\gamma$ such that $|\bar{a}| = m$ is a π-number, that is all prime divisors of m belong to π, where $\bar{a}$ is the image of a in the quotient group H/D_γ. Thus, m is the smallest integer such that $a^m = b \in D_\gamma$. Since $a \in H = A \cap d$ and d contains all roots from the set τ, we can conclude that all roots of the polynomial $X^m - b$ are contained in d. On the other hand, since $a \notin H_\gamma = A \cap d_\gamma$ and a is a root of polynomial $X^m - b$ in f_γ, we can conclude that d_γ does not contain all roots of the polynomial $X^m - b$. Since the fields d and d_γ are isomorphic and $b \in K \leq d \cap d_\gamma$, it leads to a contradiction. If $H_\gamma \backslash H \neq \emptyset$ then the same arguments lead to the same contradiction. Thus, $H = H_\gamma$.

Evidently, $c(P^\gamma) = (c(P))^\gamma$. It follows from theorem 3.3 that $(c(P))^\gamma \leq H_\gamma$ and, as $H_\gamma = H \leq B$, we can conclude that $(c(P))^\gamma \leq B$ for each $\gamma \in \Gamma$. Then $S = \langle (c(P))^\gamma \mid \gamma \in \Gamma \rangle \leq B$ and hence S is a finitely generated Γ-invariant subgroup of A which controls P. Then the assertion follows from lemma 4.3.

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